CHILDREN'S UNDERSTANDING OF FRACTIONS[1]

A COMPREENSÃO DAS CRIANÇAS SOBRE FRAÇÕES

Terezinha Nunes*
Peter Bryant**
Ursula Pretzlik, Daniel Bell, Deborah Evans e Joanna Wade***

* Professor of the Department of Education of Oxford University. Graduated in Psychology from the Federal University of Minhas Gerais and gained a PhD from City University of New York. E-mail: *terezinha.nunes@education.ox.ac.uk*

** Senior Research Fellow of the Department of Education of Oxford University. Graduated in Psychology from Cambridge University, and gained a PhD from the Unversity of London.

*** Researchers of the project "O papel da consciência na aprendizagem da matemática" (The role of awareness in the learning of mathematics), financed by the Economic and Social Research Council Teaching and Learning Research Project.

Correspondência: Department of Education. University of Oxford, 15 - Norham Gardens, Oxford, OX2 6PY.

Artigo recebido em 10/08/2008
Aprovado em 27/10/2008

Vergnaud's (1997) theory about the impact of situations on the development of mathematical concepts has inspired much research. It has already been established that problem situations in which whole numbers have different produce differences in rate of number correct responses and problem solving strategies (e.g., Carpenter & Moser, 1982; Vergnaud, 1982; 1983). Surprisingly no similar systematic comparison has been carried out in the domain of rational numbers. This paper presents two types of evidence on how situations affect children's understanding of the equivalence of fractions. Equivalence was chosen due to the central role it plays in the concept of rational numbers.

The first two sections in this paper[2] describe briefly the theoretical background for our analysis, considering the logic of fractions and a classification of situations. The subsequent sections describe the results of empirical investigations about the effect of problem solving situations on the rate of correct responses and problem solving strategies.

The importance of equivalence in the domain of rational numbers

Piaget argued that 'number is at the same time both class and asymmetrical relation' (1952, p. ix). By 'class' he meant that in order to understand cardinality, a child has to understand that all sets with 3 elements, for example, belong to the same class – i.e., are equivalent in number. Language could in principle facilitate the understanding of the logic of classes in the domain of whole numbers. However, it cannot play the same role with respect to rational numbers: in this domain, children will have to understand that 1/2, 2/4, 3/6 etc are represented by different oral and written signs but are equivalent quantities.

A classification of situations

Different classifications of situations in which rational numbers are used have been proposed in the literature, by authors such as Kieren (1988), Behr, Lesh, Post, and Silver (1983), Ohlsson (1988) and Mack (2001). However, the criteria used for classifying the situations are often not clear. In this research we will contrast two situations, part-whole and quotient situations, though we distinguish four types of situation in which rational numbers are used. Our criterion for treating situations differently is that the numerical symbols used in the situations should have different meanings.

In part-whole situations typically used with children, there is one whole – a continuous quantity – cut into equal parts; the denominator signifies the number of equal parts into which a whole was divided and the numerator signifies the number of parts taken. For example, if someone ate 3/4 of a cake, this means that the cake was divided into 4 equal parts and the person ate 3.

The most typical quotient situations used with children involve sharing continuous quantities, where the numerator represents the number of things to be shared – for example, the number of chocolate bars – and the denominator represents the number of recipients. In this situation the fraction 3/4 signifies 3 chocolate bars divided among 4 children. Note that, because 3 divided by 4 equals 3/4, this number also represents the fraction of a chocolate bar that was received by each child, irrespective of how the chocolates were cut (one could cut two chocolates in half, one in quarter, and give a half plus a quarter to each recipient).

Fractions are said to be operators when they are used in conjunction with a whole number that refers to a discrete quantity – for example, 3/4 of 24 marbles. In this case, the denominator indicates a division and the numerator indicates a multiplication; by dividing 24 by 4 and multiplying the result by 3 we find out how many marbles correspond to 3/4 of a set with 24 marbles. Note that the numbers 3 and 4 do not refer to marbles, but to groups of marbles resulting from the division of 24 marbles in 4 groups.

In spite of differences in number meanings across these three situations, they have in common the fact that they refer to extensive quantities (see Piaget, Inhelder & Szeminska, 1960, for a discussion about part-whole situations). This means that a fractional number – say 3/4 – only represents the same quantity if the whole is the same. If Paul eats 3/4 of a small cake and Mary eats 3/4 of a large cake, they eat different amounts of cake. Similarly, if 3 small chocolate bars are shared among 4 boys and 3 large chocolate bars are shared among 4 girls, the boys' and girls' portions are of different sizes. Finally, this also applies to fractions as operators: 3/4 of a set of 12 marbles and 3/4 of a set of 24 marbles refer to different numbers of marbles. Thus, a quantity-changing operation in these three situations is an operation that changes the whole.

This contrasts with situations where fractions are used as measure of an intensive quantity – for example, the probability of an event or the concentration of a mixture of water and orange juice. The probability of drawing a blue marble out of a bag with 12 marbles where 3 are white and 9 are blue is the same as the probability of drawing a blue marble out of a bag with 24 marbles where 6 are white and 18 are blue. Similarly, the concentration of orange juice is the same if we mix 1 glass of water with 3 of concentrate or if we mix 2 glasses of water with 6 of concentrate. When fractions are used as measures of intensive quantities, the differences in the size of the whole do not change the quantity: only a change in the ratio between the two quantities is a quantity-changing operation.

In order to avoid confusion, it should be pointed out that we often use fractional numbers in part-whole situations, where the whole is a conventional unit of a continuous quantity – say a meter or a kilo. In these situations, the numerator and denominator have the same meanings as in part-whole situations: 3/4 of an inch refers to the portion equivalent to dividing an inch into 4 equal parts and taking 3. This should not be confused with the use of fractions as measures of intensive quantities.

This brief contrastive analysis of situations where fractions are used identifies possible reasons for learners to perform differently across situations when solving fraction problems. However, the evidence available to date has not clarified whether children use different problem solving strategies and show different levels of success across these different situations. For reasons of space, results about only two situations are presented here. We will first present some quantitative results relative to children's understanding of equivalence and then arguments used by children in a micro-genetic study regarding the equivalence of fractions.

A quantitative survey of children's performance

We gave 130 children a fractions assessment, adapted from the CSMS Fractions 1 Paper (Hart, Brown, Kerslake, Kücherman, & Ruddock, 1985) for use with primary school children. The test includes equivalence questions presented in part-whole and quotient situations. The pupils in our study were attending three different schools in the Oxford area and were in the age range 7y9m to 10y2m. They were either in their fourth or fifth year in school. In the classroom they had been taught about fractions in the context of part-whole but not in the context of quotient situations.

The items were presented using pictures projected on a screen and also printed on the pupils' response booklets. Instructions were given orally. There were three part-whole and four quotient items. The proportion of correct responses for part-whole equivalence problems was .31 (SD=.39) and for quotient problems was .73 (SD=.37). The difference between these means was statistically significant. Performance at both age levels was better in quotient than in part-whole situations, in spite of the fact that the children had been taught about fractions in part-whole but not in quotient situations in the classroom.

These results suggested the importance of interviewing children about fractions in order to obtain more information about their problem solving strategies.

Understanding equivalence in two different situations: a qualitative analysis

We carried out a series of micro-genetic intervention studies with children in Years 4 and 5 (same age level as those in the survey) in small-group teaching sessions. By working with small groups run by a researcher outside the classroom, it was possible to observe their reactions when they face a difficulty in a judgment of equivalence and attempt to solve it following a cue given by the experimenter to draw on strategies used in part-whole or in quotient situations. The groups contained between 4 and 6 children, depending of the class size. Some of the problems we used were taken from Streefland (1997) and others were similar to his original problems but were created for this study. The children's arguments to support the idea of equivalence or non-equivalence were transcribed. The sample or arguments analysed comes from one problem involving the equivalence between 1/3 and 2/6. This problem was presented in the second session with the researcher. In the first session the children had already solved a problem about sharing 3 chocolates among 4 children and indicating what fraction of a chocolate bar is received by each child. The problem in the second session was: Six children go to a pizzeria and order two pizzas. The waiter first brings one, and then the other pizza. How could they share the first pizza? And the second? What fraction of pizza does each child receive at the end? Could they share it differently if the waiter brought the two pizzas at the same time? What fraction would each child have? Do they eat the same amount if they share in these different ways?

The dialogue below illustrates how the problem worked in the session. The letter R designates the researcher's turn, the other letters designate the children's turns.

R - What fraction of the first pizza do they receive?

M - One sixth.

R - Why is that?

M - Because there are six children so they split the pizza in sixths [no marks were made on the drawing]. …

R - If they get one sixth from that one and one sixth from that one, how many sixths do they have altogether?

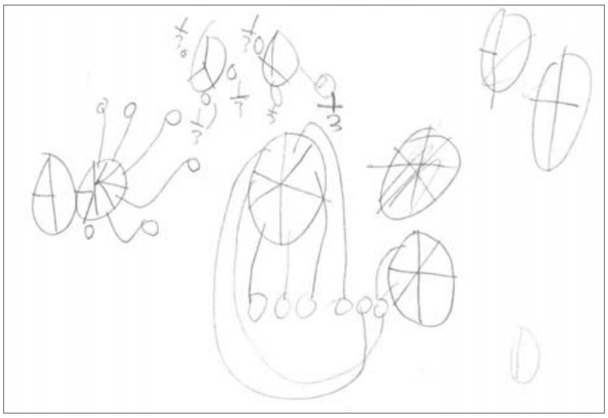
St - Two sixths. …

R - [if the waiter brought the pizzas at the same time], how would they share them differently, what are the two ways that they can share it out?

G - They can share it in thirds.

ST - Those get a third from that one, and those three get a third from that one.

When the pupils were asked whether the children would receive the same amount of pizza, most had no doubt that the children would eat the same amount of pizza if they had 2/6 of 1/3. The equivalence of these fractions seemed so obvious that sometimes it was difficult to elicit verbal arguments. Some justifications did appear when the children were prompted to explain their reasoning better.



We classified these arguments into different types, presented below with illustrative examples. It should be noted that they were quite often verbal arguments added to the demonstration by means of drawings that there were appropriate correspondences between the number of pizzas and the number of sharers (see Figure 1). Thus the pupils in these cases were using correspondences between pizzas and children to formulate their arguments.

Figure 1 shows the drawings produced by a child who initially attempted to establish visual comparisons between 2/6 and 1/3 (on the right). She then changed her approach and established correspondences between the pizzas and the children, represented by the small circles. She then concluded: "If they have two pizzas, then they could give the first pizza to three girls and then the next one to another three girls. (…) If they all get one piece of that each, and they all get the same amount, they all get the same amount"

Exhaustive and fair division

C: Because it wouldn't really matter when they shared it, because when they shared it in three, those three get it and that pizza is gone, and those three share this, and that pizza is gone. When they shared it at the same time, they share it fairly and the pizzas are gone. [The argument seems to be based on the idea of exhaustive and fair division of the same whole; the child indicates the correspondences throughout without making drawings]

Numerical arguments without

H: Because one third is a third of three and two sixths is a third of six.[This justification seems to be based on numerical relations without reference to the context]

An algebraic argument

P (wrote 2/6 + 2/6 + 2/6 = 6 and 1/3 plus 1/3 plus 1/3 = 3): There's two sixths [pointing to the first 2/6 on the page], add two sixths three times to make six sixths. With one third, you need to add one third three times to make three thirds. [This is an algebraic argument based on the composition of parts: if a+a+a=b+b+b, then a and b must be the same irrespective of what you name them].

Scalar reasoning

C: I put it [the pizza] into thirds and I put the girls in half [her gestures seemed to indicate that she took half of the pizzas for half of the children, appearing to use scalar reasoning].

Some children attempted to use a partitioning strategy, thus changing the meaning of the numbers into those that are used in part-whole situations. In these cases, they drew the pizzas (sometimes replaced with rectangles), divided each into either 3 or 6 parts, and marked 1/3 and 2/6 on the pizzas, respectively. This perceptual strategy invariably left them in doubt about whether 1/3 and 2/6 are the same.

Conclusions

The research briefly outlined here illustrates how the meaning of fractions differs across situations and how these differences can affect children's strategies and arguments when they are trying to decide whether two fractions are equivalent. Not only the pupils' level of success differs across part-whole and quotient situations when they are asked to judge the equivalence of different fractions but also the ideas they explored when analysing equivalence support the need to distinguish between the situations. For reasons of space, it was not possible to discuss the methods or results in detail. However, we hope to have provided sufficient information to show that it is urgent that further systematic comparisons between the different situations that give meaning to fractions be carried out.

References

Behr, M., Lesh, R., Post, T., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), Acquisition of Mathematical Concepts and Processes (pp. 91-126). New York: Academic Press.

Carpenter, T. P., & Moser, J. M. (1982). The development of addition and subtraction problem solving. In T. P. Carpenter & J. M. Moser & R. T. A (Eds.), Addition and subtraction: A cognitive perspective (pp. 10-24). Hillsdale (NJ): Lawrence Erlbaum.

Hart, K., Brown, M., Kerslake, D., Küchermann, D., & Ruddock, G. (1985). Chelsea Diagnostic Mathematics Tests. Teacher's Guide. Windsor (UK): NFER-Nelson.

Kieren, T. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), Number concepts and operations in the middle-grades (pp. 53-92). Reston (VA): National Council of Teachers of Mathematics.

Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. Journal for Research in Mathematics Education, 32, 267-295.

Ohlsson, S. (1988). Mathematical meaning and applicational meaning in the semantics of fractions and related concepts. In J. Hiebert & M. Behr (Eds.), Number Concepts and Operations in the Middle Grades (pp. 53-92). Reston (VA): National Council of Mathematics Teachers.

Piaget, J. (1952). The Child's Conception of Number. London: Routledge & Kegan Paul.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). The Child's Conception of Geometry. New York: Harper & Row.

Streefland, L. (1997). Charming fractions or fractions being charmed? In T. Nunes & P. Bryant (Eds.), Learning and Teaching Mathematics. An International Perspective (pp. 347-372). Hove (UK): Psychology Press.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter & J. M. Moser & R. T. A (Eds.), Addition and subtraction: A cognitive perspective (pp. 60-67). Hillsdale (NJ): Lawrence Erlbaum.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), Acquisition of mathematics concepts and processes (pp. 128-175). London: Academic Press.

Vergnaud, G. (1997). The nature of mathematical concepts. In T. Nunes & P. Bryant (Eds.), Learning and Teaching Mathematics. An International Perspective (pp. 1-28). Hove (UK): Psychology Press.

Notes

[1] Editor's Note: A version of this text was previously published in Nunes, T., Bryant, P., Pretzlik, U., Bell, D., Evans, D., & Wade, J. (2007). La compréhension des fractions chez les enfants. In M. Merri (Ed.), Activité humaine et conceptualisation. Questions à Gérard Vergnaud (pp. 255-262). Toulouse, France: Presses Universitaires du Mirail.

[2] This project was supported by the ESRC Teaching and Learning Research Programme. T. Nunes' time on the project was financed by a British Academy Research Readership. We are thankful for the support provide by these institutions and also very grateful to the teachers and pupils who generously gave so much of their time.